# Separable solutions of quasilinear Lane-Emden equations

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**Abstract** For $0 < p-1 < q$ and either $\epsilon = 1$ or $\epsilon = -1$, we prove the existence of solutions of $-\Delta_p u = \epsilon u^q$ in a cone $C_S$, with vertex 0 and opening $S$, vanishing on $\partial C_S$, under the form $u(x) = |x|^{-\beta}\omega(\frac{x}{|x|})$. The problem reduces to a quasilinear elliptic equation on $S$ and existence is based upon degree theory and homotopy methods. We also obtain a non-existence result in some critical case by an integral type identity.

*2010 Mathematics Subject Classification.* 35J92, 35J60, 47H11, 58C30.

*Key words.* quasilinear elliptic equations; $p$-Laplacian; cones; Leray-Schauder degree.

## 1 Introduction

It is well established that the description of the boundary behavior of positive singular solutions of Lane-Emden equations

$$-\Delta u = \epsilon u^q \tag{1.1}$$

with $q > 1$ in a domain $\Omega \subset \mathbb{R}^N$ is greatly helped by using specific separable solutions of the same equation. This was performed in 1991 by Gmira-Véron [8] in the case $\epsilon = -1$ and more recently by Bidaut-Véron-Ponce-Véron [3] in the case $\epsilon = 1$. If the domain is assumed to be a cone $C_S = \{x \in \mathbb{R}^N \setminus \{0\} : x/|x| \in S\}$ with vertex 0 and opening $S \subsetneq S^{N-1}$ (the unit sphere in $\mathbb{R}^N$), separable solutions of (1.1) vanishing on $\partial C_S \setminus \{0\}$ were of the form

$$u(x) = |x|^{-\frac{2}{q-1}}\omega(x/|x|), \tag{1.2}$$

with $\omega$ satisfying

$$-\Delta'\omega - \ell_{q,N}\omega - \epsilon\omega^q = 0 \qquad \text{in } S, \tag{1.3}$$

vanishing on $\partial S$ and where $\ell_{q,N} = \left(\left(\frac{2}{q-1}\right)\left(\frac{2q}{q-1}-N\right)\right)$ and $\Delta'$ is the Laplace-Beltrami operator on $S^{N-1}$. To this equation is associated the functional

$$J(\phi) := \int_S \left(\frac{1}{2}|\nabla'\phi|^2 - \frac{\ell_{q,N}}{2}\phi^2 - \frac{\epsilon}{q+1}|\phi|^{q+1}\right) dv_g, \tag{1.4}$$

where $\nabla'$ is the covariant derivative on $S^{N-1}$. In the case $\epsilon = 1$, non-existence of a non-trivial positive solution of (1.3) when $\ell_{q,N} \geq \lambda_{_S}$ (the first eigenvalue of $-\Delta'$ in $W_0^{1,2}(S)$) follows by multiplying the equation by the first eigenfunction and integrating over $S$; existence holds when $\ell_{q,N} < \lambda_{_S}$ and $q < \frac{N+1}{N-3}$ by classical variational methods, and again non-existence holds when $q \geq \frac{N+1}{N-3}$ and $S \subset S_+^{N-1}$ is starshaped by using an integral identity [3, Th 2.1,Cor 2.1]. When $\epsilon = -1$, non-existence of a non-trivial solution of (1.3) when $\ell_{q,N} \leq \lambda_{_S}$ is obtained by multiplying the equation by $\omega$ and integrating over $S$, while existence when $\ell_{q,N} > \lambda_{_S}$ follows by minimizing $J$ over $W_0^{1,2}(S) \cap L^{q+1}(S)$.

In this paper we investigate similar questions for the quasilinear Lane-Emden equations

$$-\,div\left(|\nabla u|^{p-2}\nabla u\right) = \epsilon u^q \qquad \text{in } C_S, \tag{1.5}$$

where $S$ is a smooth subset of $S^{N-1}$, $q > p-1 > 0$ and $\epsilon = \pm 1$ and we look for positive solutions $u$, vanishing on $\partial C_{_S} \setminus \{0\}$, under the separable form

$$u(x) = |x|^{-\beta}\omega(x/|x|). \tag{1.6}$$

It is straightforward to check that $u$ is a solution of (1.5) provided

$$\beta = \beta_q := \frac{p}{q+1-p} \tag{1.7}$$

and $\omega$ is a positive solution of

$$-div\left(\left(\beta_q^2\omega^2 + |\nabla'\omega|^2\right)^{(p-2)/2}\nabla'\omega\right) - \beta_q\lambda(\beta_q)\left(\beta_q^2\omega^2 + |\nabla'\omega|^2\right)^{(p-2)/2}\omega = \epsilon\omega^q \tag{1.8}$$

in $S$ vanishing on $\partial S$, where $div(\cdot)$ is the divergence operator defined according to the intrinsic metric $g$ and where we have set

$$\lambda(\beta) = \beta(p-1) + p - N. \tag{1.9}$$

If $\epsilon = 0$, it is now well-known that positive p-harmonic functions in $C_S$ vanishing on $\partial C_S$ exist under the form (1.6), and either they are regular at 0 and $\beta = -\tilde{\beta}_{_S} < 0$, or they are singular and $\beta = \beta_{_S} > 0$, where the values of $\tilde{\beta}_{_S}$, $\beta_S$ are unique. In this case $\omega = \tilde{\omega}_{_S}$ or $\omega_{_S}$ is a solution of

$$-\,div\left(\left(\beta^2\omega^2 + |\nabla'\omega|^2\right)^{(p-2)/2}\nabla'\omega\right) - \beta\lambda(\beta)\left(\beta^2\omega^2 + |\nabla'\omega|^2\right)^{(p-2)/2}\omega = 0 \tag{1.10}$$

in $S$, where $\beta = \tilde{\beta}_{_S}$ or $\beta_{_S}$. The existence of $(\tilde{\beta}_{_S}, \tilde{\omega}_{_S})$ is due to Tolksdorf in a pioneering work [18]. Tolksdorf's method has been adapted by Véron [20] in order

to prove the existence of $(\beta_S, \omega_S)$. Later on Porretta and Véron [13] obtained a more general proof of the existence of such couples. Notice that $\beta_S$ (as well as $\tilde{\beta}_S$) is uniquely determined while $\omega$ is unique up to homothety. In both cases the proofs rely on strong maximum principle.

When $p \neq 2$, existence of a nontrivial solution in the case $\epsilon = 1$ is obtained in [2] when $N = 2$ and $\beta_q < \beta_S$ by a dynamical system approach; while if $\epsilon = -1$ and $\beta_q > \beta_S$, such an existence is proved in [20] by a suitable adaptation of Tolksdorf's construction. Notice that no functional can be associated to (1.8), excepted in the case $q = q^* = \frac{Np}{N-p} - 1$. In such a case (1.8) is the Euler-Lagrange equation for the functional

$$J_q(\phi) := \int_S \left( \frac{1}{p} \left( \beta_{q^*}^2 \phi^2 + |\nabla' \phi|^2 \right)^{\frac{p}{2}} - \frac{\epsilon}{q^* + 1} |\phi|^{q^*+1} \right) dv_g, \tag{1.11}$$

and existence of a non-trivial solution of (1.8) with $\epsilon = 1$ is derived from the mountain pass theorem. In all the other cases variational techniques cannot be used and have to be replaced by topological methods based upon Leray-Schauder degree. Define $q_c$ by

$$q_c = q_{c,p} = \begin{cases} \frac{(N-1)p}{N-1-p} - 1 & \text{if } p < N-1 \\ \infty & \text{if } p \geq N-1, \end{cases}$$

then we prove the following results:

**I** *Let $\epsilon = 1$. Assume $p > 1$, $q < q_c$ and $\beta_q < \beta_S$, then (1.8) admits a positive solution in $S$ vanishing on $\partial S$.*

**II** *Let $\epsilon = -1$. Assume $p > 1$ and $\beta_q > \beta_S$, then (1.8) admits a unique positive solution in $S$ vanishing on $\partial S$.*

The result **I** is based upon sharp Liouville theorems for solutions of (1.5) in $\mathbb{R}^N$ or $\mathbb{R}^N_+$ respectively due to Serrin-Zou [17] and Zou [23]. In the case of **II**, the existence part is already known, but we give here a simpler form than the one in [20], using a topological deformation acting on the exponent $p$. In the case $\epsilon = 1$, the result is optimal in the case $q = q_c$; indeed, using an integral identity, we also prove

**III** *Let $\epsilon = 1$, $S \subsetneq S^{N-1}_+$ be a starshaped domain and $1 < p < N-1$. If $q = q_c$, then (1.8) admits no positive solution in $S$ vanishing on $\partial S$.*

Notice that when $p = 2$ an integral identity was used in [3] to prove non existence for all $q \geq q_{c,2}$. The form which is derived in the case $p \neq 2$ is much more complicated and we prove non-existence only in the case $q = q_{c,p}$.

Finally, the constraint $\beta_q < \beta_S$ in **I** (respectivey, $\beta_q > \beta_S$ in **II**) is sharp. When $\epsilon = 1$, the non-existence of positive solutions of (1.8) when $\beta_q \geq \beta_S$ has been proved in [2]. The method is based upon strong maximum principle. When $\epsilon = -1$ a somewhat similar method is used in [22] and yields to non-existence results when $\beta_q \leq \beta_S$. Notice that the obtention of such results when $p = 2$ is straightforward.

## 2 Nonexistence for the reaction problem

Let $S$ be a bounded $C^2$ sub-domain of $S^{N-1}$. We consider the positive solutions in $S$ of

$$-\,div\left(\left(\beta^2\omega^2+|\nabla'\omega|^2\right)^{(p-2)/2}\nabla'\omega\right)-\beta\lambda(\beta)\left(\beta^2\omega^2+|\nabla'\omega|^2\right)^{(p-2)/2}\omega=\omega^q \quad (2.1)$$

vanishing on $\partial S$. Recall that $\lambda(\beta)$ is given by (1.9) and that, in connection with problem (1.5), we have interest in the special case where $\beta=\beta_q$ is given by (1.7). The following Pohozaev-type identity, which is valid for any $\beta$, is the key for non-existence. We denote by $S_+^{N-1}$ the half sphere.

**Proposition 2.1** *Let $S \subsetneq S^{N-1}$ be a $C^2$ domain and $\phi$ the first eigenfunction of $-\Delta'$ in $W_0^{1,2}(S_+^{N-1})$. If $\omega \in W_0^{1,p}(S)\cap C(\overline{S})$ is a positive solution in $S$ of (2.1), and if we set $\Omega=(\beta^2\omega^2+|\nabla'\omega|^2)^{1/2}$, then the following identity holds*

$$\left(1-\frac{1}{p}\right)\int_{\partial S}|\omega_\nu|^p\phi_\nu dS = A\int_S \omega^{q+1}\phi\, d\sigma + B\int_S \Omega^{p-2}|\nabla'\omega|^2\phi\, d\sigma + C\int_S \Omega^{p-2}\omega^2\phi\, d\sigma, \quad (2.2)$$

*with*

$$A=A(\beta):=-\frac{N-1}{q+1}-\beta(p\beta+p-N) \quad (2.3)$$

$$B=B(\beta):=\frac{N-1-p}{p}+\beta(p\beta+p-N), \quad (2.4)$$

$$C=C(\beta):=\beta^2\left(\frac{N-1}{p}-(p\beta+p-N)\lambda(\beta)\right). \quad (2.5)$$

In order to prove Proposition 2.1, we start with the following lemma.

**Lemma 2.1** *Let $S\subset S^{N-1}$ be a $C^2$ domain and $\phi\in C^2(\overline{S})$. If $\omega\in W_0^{1,p}(S)\cap C(\overline{S})$ is a positive solution of (2.1) in $S$, we have:*

$$\begin{aligned}\left(1-\frac{1}{p}\right)\int_{\partial S}|\omega_\nu|^p\phi_\nu dS &= \int_S\left(\frac{\Delta'\phi}{q+1}-\beta(p\beta+p-N)\phi\right)\omega^{q+1}\,d\sigma-\frac{1}{p}\int_S\Omega^p\Delta'\phi\,d\sigma\\ &+\int_S\Omega^{p-2}D^2\phi(\nabla'\omega,\nabla'\omega)d\sigma+\beta(p\beta+p-N)\int_S\Omega^{p-2}|\nabla'\omega|^2\phi\,d\sigma\\ &-\beta^2(p\beta+p-N)\lambda(\beta)\int_S\Omega^{p-2}\omega^2\phi\,d\sigma.\end{aligned} \quad (2.6)$$

*Proof.* By the regularity theory of $p$-Laplace type equations (see e.g. [6], [19] and the Appendix in [13]) it turns out that $\omega\in C^{1,\gamma}(\overline{S})$ for some $\gamma\in(0,1)$, and since $\left(\beta^2\omega^2+|\nabla'\omega|^2\right)>0$ in the interior, by elliptic regularity we have $\omega\in C^2(S)$. Let $\phi\in C^2(S)$ be a given function and $\zeta\in C_c^1(S)$; since $\zeta$ is compactly supported we

can multiply (2.1) by the test function $\langle\nabla'\omega, \nabla'\phi\rangle\zeta$. Integrating by parts we get (using the notation $\Omega := (\beta^2\omega^2 + |\nabla'\omega|^2)^{1/2}$)

$$\int_S \Omega^{p-2}\left(\frac{1}{2}\langle\nabla'|\nabla'\omega|^2, \nabla'\phi\rangle + D^2\phi(\nabla'\omega, \nabla'\omega)\right)\zeta\, d\sigma + \int_S \Omega^{p-2}\langle\nabla'\omega, \nabla\zeta\rangle\,\langle\nabla'\omega, \nabla'\phi\rangle\, d\sigma$$
$$= \beta\lambda(\beta)\int_S \Omega^{p-2}\omega\langle\nabla'\omega, \nabla'\phi\rangle\,\zeta\, d\sigma + \frac{1}{q+1}\int_S \langle\nabla'\omega^{q+1}, \nabla'\phi\rangle\,\zeta\, d\sigma.$$

Since

$$\Omega^{p-2}\,\frac{1}{2}\langle\nabla'|\nabla'\omega|^2, \nabla'\phi\rangle = \frac{1}{p}\langle\nabla'\Omega^p, \nabla\phi\rangle - \beta^2\Omega^{p-2}\omega\langle\nabla'\omega, \nabla'\phi\rangle$$

we obtain, due to (1.9),

$$\frac{1}{p}\int_S \langle\nabla'\Omega^p, \nabla'\phi\rangle\,\zeta\, d\sigma + \int_S \Omega^{p-2}\, D^2\phi(\nabla'\omega, \nabla'\omega)\,\zeta\, d\sigma + \int_S \Omega^{p-2}\langle\nabla'\omega, \nabla'\zeta\rangle\,\langle\nabla'\omega, \nabla'\phi\rangle\, d\sigma$$
$$= \beta(p\beta + p - N)\int_S \Omega^{p-2}\omega\langle\nabla'\omega, \nabla'\phi\rangle\,\zeta\, d\sigma + \frac{1}{q+1}\int_S \langle\nabla'\omega^{q+1}, \nabla'\phi\rangle\,\zeta\, d\sigma.$$

Integrating by parts the first and last term we get

$$-\frac{1}{p}\int_S \Omega^p\langle\nabla'\phi, \nabla'\zeta\rangle\, d\sigma + \frac{1}{q+1}\int_S \omega^{q+1}\langle\nabla'\phi, \nabla'\zeta\rangle\, d\sigma + \int_S\left(\frac{\omega^{q+1}}{q+1} - \frac{\Omega^p}{p}\right)\Delta'\phi\,\zeta\, d\sigma$$
$$+ \int_S \Omega^{p-2}\, D^2\phi(\nabla'\omega, \nabla'\omega)\,\zeta\, d\sigma + \int_S \Omega^{p-2}\langle\nabla'\omega, \nabla'\zeta\rangle\,\langle\nabla'\omega, \nabla'\phi\rangle\, d\sigma$$
$$= \beta(p\beta + p - N)\int_S \Omega^{p-2}\omega\langle\nabla'\omega, \nabla'\phi\rangle\,\zeta\, d\sigma. \tag{2.7}$$

Now we choose $\zeta = \zeta_\delta$, where $\zeta_\delta$ is a sequence of $C^1$ compactly supported functions such that $\zeta_\delta(\sigma) \to 1$ for every $\sigma \in S$ and $|\nabla'\zeta_\delta|$ is bounded in $L^1(S)$. It is easy to see by integration by parts that we have for every continuous vector field $F \in C(\overline{S})$

$$\int_S \langle F, \nabla'\zeta_\delta\rangle\, d\sigma \to -\int_{\partial S}\langle F, \nu(\sigma)\rangle\, d\sigma$$

where $\nu$ is the outward unit normal on $\partial S$. We take $\zeta = \zeta_\delta$ in (2.7) and we let $\delta \to 0$. Using that $\omega \in C^1(\overline{S})$ and that, by Hopf lemma, $\omega_\nu := \langle\nabla'\omega, \nu(\sigma)\rangle < 0$ we can actually pass to the limit in the integrals containing $\nabla'\zeta_\delta$. Recalling that $\omega = 0$ and $\nabla'\omega = -|\omega_\nu|\nu$ on $\partial S$ we obtain

$$\left(1 - \frac{1}{p}\right)\int_{\partial S}|\omega_\nu|^p\phi_\nu dS = \int_S\left(\frac{\omega^{q+1}}{q+1} - \frac{\Omega^p}{p}\right)\Delta'\phi\, d\sigma + \int_S \Omega^{p-2}D^2\phi(\nabla'\omega, \nabla'\omega)d\sigma$$
$$- \beta(p\beta + p - N)\int_S \Omega^{p-2}\omega\langle\nabla'\omega, \nabla'\phi\rangle d\sigma. \tag{2.8}$$

Multiplying (2.1) by $\omega\phi$ we derive

$$\int_S \Omega^{p-2}\omega\langle\nabla'\omega, \nabla'\phi\rangle d\sigma = -\int_S \Omega^{p-2}|\nabla'\omega|^2\phi\, d\sigma + \beta\lambda(\beta)\int_S \Omega^{p-2}\omega^2\phi\, d\sigma + \int_S \omega^{q+1}\phi\, d\sigma,$$

so that (2.8) becomes, replacing its last term,

$$\left(1-\frac{1}{p}\right)\int_{\partial S}|\omega_\nu|^p\phi_\nu dS=\int_S\left(\frac{\omega^{q+1}}{q+1}-\frac{\Omega^p}{p}\right)\Delta'\phi\,d\sigma+\int_S\Omega^{p-2}D^2\phi(\nabla'\omega,\nabla'\omega)d\sigma$$
$$-\beta(p\beta+p-N)\int_S\omega^{q+1}\,\phi\,d\sigma+\beta(p\beta+p-N)\int_S\Omega^{p-2}|\nabla'\omega|^2\phi$$
$$-\beta^2(p\beta+p-N)\lambda(\beta)\int_S\Omega^{p-2}\omega^2\phi\,d\sigma.$$

which is (2.6). □

*Proof of Proposition 2.1.* We use Lemma 2.1 choosing in (2.6) $\phi$ to be the first eigenfunction of $-\Delta'$ in $W_0^{1,2}(S_+^{N-1})$. Since $\Delta'\phi=(1-N)\phi$, $D^2\phi=-\phi g_0$, we get

$$\begin{aligned}\left(1-\frac{1}{p}\right)\int_{\partial S}|\omega_\nu|^p\phi_\nu dS=&-\int_S\left(\frac{N-1}{q+1}+\beta(p\beta+p-N)\right)\omega^{q+1}\phi\,d\sigma\\&+\frac{N-1}{p}\int_S\Omega^p\phi\,d\sigma-\int_S\Omega^{p-2}|\nabla'\omega|^2\,\phi\,d\sigma\\&+\beta(p\beta+p-N)\int_S\Omega^{p-2}|\nabla'\omega|^2\phi\,d\sigma\\&-\beta^2(p\beta+p-N)\lambda(\beta)\int_S\Omega^{p-2}\omega^2\phi.\end{aligned}\tag{2.9}$$

Then, using also the definition of $\Omega$, (2.2) follows, with $A$, $B$ and $C$ given by (2.3)-(2.5). □

We shall say that a $C^2$ domain $S\subset S_+^{N-1}$ is starshaped if there exists a spherical harmonic $\phi$ of degree 1 such that $\phi>0$ on $S$ and for any $a\in\partial S$,

$$\langle\nabla\phi,\nu_a\rangle\leq 0\tag{2.10}$$

where $\nu_a$ is the unit outward normal vector to $\partial S$ at $a$ in the tangent plane $T_a$ to $S^{N-1}$. It also means that there exists some $x_0\in S$ such that the geodesic connecting $x_0$ and $a$ remains inside $S$.

**Theorem 2.1** *Assume that $1<p<N-1$, $q=q_c$ and $S\subset S_+^{N-1}$ is starshaped. Then (2.1) admits no positive solution in $S$ vanishing on $\partial S$.*

*Proof.* Recall that in (1.8) we have $\beta_q=\frac{p}{q-(p-1)}$, hence different values of $q$ are in one-to-one correspondence with different values of $\beta$. We first notice that, if $q=q_c$ the corresponding critical $\beta$ is given by

$$\beta_c:=\frac{p}{q_c-(p-1)}=\frac{N-1-p}{p}\,.\tag{2.11}$$

We use now Proposition 2.1 with $\beta = \beta_q$ and we analyze the values of the coefficients $A$, $B$, $C$ given by (2.3)-(2.5) as functions of $\beta$. First of all, since $q+1 = \frac{p(1+\beta)}{\beta}$, we have

$$A = -\frac{(N-1)\beta}{p(1+\beta)} - \beta(p\beta + p - N) = -\frac{\beta}{(\beta+1)}\left(\frac{N-1}{p} + p(\beta+1)^2 - N(\beta+1)\right)$$

and since from (2.11) we have $\beta_c + 1 = \frac{N-1}{p}$ we deduce

$$A = -\frac{\beta}{(\beta+1)}\, p\left(\beta + 1 - \frac{1}{p}\right)(\beta - \beta_c)\,.$$

Still using (2.11), we also get

$$B = \beta_c + \beta(p(\beta - \beta_c) - 1) = (\beta - \beta_c)(\beta p - 1)\,.$$

Finally, using (1.9) and (2.11) we have

$$\begin{aligned} C &= \beta^2\left(\frac{N-1}{p} - (p\beta + p - N)((p-1)\beta + p - N)\right) \\ &= \beta^2\left(\beta_c + 1 - (p(\beta - \beta_c) - 1)(p(\beta - \beta_c) - (\beta + 1))\right) \\ &= \beta^2(\beta - \beta_c)(1-p)\left(p\beta - 1 - \tfrac{p(N-p)}{p-1}\right). \end{aligned} \tag{2.12}$$

Therefore $A \geq 0$, $B \geq 0$ and $C \geq 0$ can be obtained only if $q = q_c$, i.e. $\beta = \beta_c$, in which case $A = B = C = 0$. Since $\phi_\nu \leq 0$ because $S$ is star–shaped, we deduce from (2.2) that $|\omega_\nu|^p \phi_\nu = 0$ on $\partial S$. Unless $\omega$ is identically zero, we have $\omega_\nu < 0$ by Hopf boundary lemma. Then $\phi_\nu \equiv 0$, and using the equation satisfied by $\phi$ and Gauss formula, we derive

$$\lambda_S \int_S \phi\, d\sigma = 0 \Longrightarrow \phi \equiv 0 \qquad \text{in } S,$$

which is impossible since $\phi > 0$ in $S_+^{N-1}$. This proves the first assertion. □

*Remark.* If $p = 2$, it is proved in [3] that the nonexistence result of Theorem 2.1 holds for every $q \geq q_c$, which suggests that our result above is not optimal. The proof in [3] cannot be applied here since the term $\int_S \Omega^{p-2}\omega\langle\nabla'\omega, \nabla'\phi\rangle d\sigma$ is completely integrable only if $p = 2$. However, we conjecture that, even when $p \neq 2$, the conclusion of Theorem 2.1 holds under the more general condition $q \geq q_c$.

*Remark.* If we assume that $p \neq 2$, the proof of Theorem 2.1 relies on the existence of a positive function $\phi$ in $S$, satisfying (2.10) on $\partial S$ and

$$\frac{\Delta'\phi}{(q+1)\phi} - \beta(p\beta + p - N) \geq 0, \tag{2.13}$$

$$\frac{pD^2\phi(\xi,\xi) - \Delta'\phi}{p\phi} + \beta(p\beta + p - N) \geq 0 \quad \forall \xi \in S^{N-1}, \tag{2.14}$$

and

$$-\frac{\Delta'\phi}{p\phi} - (p\beta + p - N)((p-1)\beta + p - N) \geq 0. \tag{2.15}$$

**Remark 2.1** For the sake of completeness, we recall the non-existence result obtained in [2, Th 1]:

*Let $\epsilon = 1$ and $0 < p-1 < q$. If $\beta_q \geq \beta_S$, there exists no positive solution of (1.8) in $S$ which vanishes on $\partial S$.*

# 3 Existence for the reaction problem

Concerning the problem with reaction we consider a more general statement than Theorem I, replacing the sphere by a complete $d$-dimensional Riemannian manifold $(M, g)$ and suppose that $S$ is a relatively compact smooth open domain of $M$. We denote by $\nabla := \nabla_g$ the gradient of a function identified with its covariant derivatives and by $div := div_g$ the intrinsic divergence operator acting on vector fields. The following result is proved in [13].

**Theorem 3.1** *For any $\beta > 0$ there exists a unique $\Lambda_\beta > 0$ and a unique (up to an homothety) positive function $\omega_\beta \in C^2(S) \cap C^1(\overline{S})$ solution of*

$$\begin{cases} -div\left(\left(\beta^2\omega_\beta^2 + |\nabla\omega_\beta|^2\right)^{\frac{p-2}{2}}\nabla\omega_\beta\right) = \beta\Lambda_\beta\left(\beta^2\omega_\beta^2 + |\nabla\omega_\beta|^2\right)^{\frac{p-2}{2}}\omega_\beta \quad \text{in } S \\ \omega_\beta = 0 \quad \text{on } \partial S. \end{cases} \tag{3.1}$$

*The mapping $\beta \mapsto \Lambda_\beta$ is continuous and decreasing, and the spectral exponent $\beta_{\scriptscriptstyle S}$ is the unique $\beta > 0$ such that $\Lambda_{\beta_{\scriptscriptstyle S}} = \beta_{\scriptscriptstyle S}(p-1) + p - d - 1$.*

**Remark 3.1** *Let us notice that the monotone character of $\beta \mapsto \Lambda_\beta$ implies that*

$$0 < \beta < \beta_{\scriptscriptstyle S} \Longleftrightarrow \Lambda_\beta - \beta(p-1) > \Lambda_{\beta_{\scriptscriptstyle S}} - \beta_{\scriptscriptstyle S}(p-1) = p - d - 1$$

*Therefore, if we set $\lambda(\beta) = \beta(p-1) + p - d - 1$, we deduce that*

$$0 < \beta < \beta_{\scriptscriptstyle S} \Longleftrightarrow \Lambda_\beta > \lambda(\beta). \tag{3.2}$$

Let us now prove the existence of solutions for the reaction problem.

**Theorem 3.2** *Assume $1 < p < d$ and $p - 1 < q < q_c := pd/(d-p) - 1$. Then for any $0 < \beta < \beta_{\scriptscriptstyle S}$, there exists a positive function $\omega \in C(\overline{S}) \cap C^2(S)$ satisfying*

$$\begin{cases} -div\left((\beta^2\omega^2 + |\nabla\omega|^2)^{\frac{p}{2}-1}\nabla\omega\right) = \beta\lambda(\beta)(\beta^2\omega^2 + |\nabla\omega|^2)^{\frac{p}{2}-1}\omega + \omega^q \quad \text{in } S \\ \omega = 0 \quad \text{on } \partial S, \end{cases} \tag{3.3}$$

*where $\lambda(\beta) = \beta(p-1) + p - d - 1$.*

In order to prove Theorem 3.2, we use topological arguments as it is often needed in a non-variational setting. In particular, following a strategy similar as in [15], our proof is based upon the following fixed point theorem which is only one possible consequence of Leray–Schauder degree theory to compute the fixed point index of compact mappings. Such results were developed mostly by Krasnoselskii ([9]), we refer to Proposition 2.1 and Remark 2.1 in [5] for the statement below.

**Theorem 3.3** *Let $X$ be a Banach space and $K \subset X$ a closed cone with non empty interior. Let $F : K \times \mathbb{R}_+ \to K$ be a compact mapping, and let $\Phi(u) = F(u, 0)$ (compact mapping from $K$ into $K$). Assume the following holds: there exist $R_1 < R_2$ and $T > 0$ such that*
*(i) $u \neq s\Phi(u)$ for every $s \in [0, 1]$ and every $u$: $\|u\| = R_1$.*
*(ii) $F(u, t) \neq u$ for every $(u, t)$: $\|u\| \leq R_2$ and $t \geq T$.*
*(iii) $F(u, t) \neq u$ for every $u$: $\|u\| = R_2$ and every $t \geq 0$.*
*Then, the mapping $\Phi$ has a fixed point $u$ such that $R_1 < \|u\| < R_2$.*

We also recall the following non-existence results respectively due to Serrin and Zou [17], and Zou [23].

**Theorem 3.4** *Assume $1 < p < d$ and $p - 1 < q < q_c$. Then there exists no $C^1$ positive solution of*

$$-\Delta_p u = u^q \tag{3.4}$$

*in $\mathbb{R}^d$.*

**Theorem 3.5** *Assume $1 < p < d$ and $p - 1 < q < q_c$. Then there exists no $C^1$ positive solution of*

$$-\Delta_p u = u^q \tag{3.5}$$

*in $\mathbb{R}^d_+ := \{x = (x_1, ..., x_d) : x_d > 0\}$ vanishing on $\partial\mathbb{R}^d_+ := \{x = (x_1, ..., x_d) : x_d = 0\}$.*

*Proof of Theorem 3.2.* Define the operator $\mathcal{A}$ in $W_0^{1,p}(S)$ as

$$\mathcal{A}(\omega) := -div_g \left( (\beta^2\omega^2 + |\nabla\omega|^2)^{\frac{p}{2}-1} \nabla\omega \right) + \beta^2 \, \omega(\beta^2\omega^2 + |\nabla\omega|^2)^{\frac{p}{2}-1} .$$

Note that $\mathcal{A}$ is the derivative of the functional

$$J(w) = \frac{1}{p} \int_S (\beta^2\omega^2 + |\nabla\omega|^2)^{\frac{p}{2}} \, dv_g$$

Since $J$ is strictly convex, then $\mathcal{A}$ is a strictly monotone operator from $W_0^{1,p}(S)$ into $W^{-1,p'}(S)$, henceforth its inverse is well defined and continuous [10]. In order to apply Theorem 3.3, we denote by $X = C_0^1(\overline{S})$, the closure of $C_0^1(S)$ in $C^1(\overline{S})$. Clearly $X \subset W_0^{1,p}(S)$, with continuous imbedding, if it is endowed with its natural norm $||.||_X := ||.||_{C^1(\overline{S})}$. Furthermore, since $\partial S$ is $C^2$, $C^1(\overline{S}) \cap W_0^{1,p}(S) = C_0^1(\overline{S})$. If

$K$ is the cone of nonnegative functions in $S$, it has a nonempty interior. For $t > 0$, we set

$$F(\omega, t) := \mathcal{A}^{-1}\left(\beta\,(\lambda(\beta) + \beta + t)\,\omega(\beta^2\omega^2 + |\nabla\omega|^2)^{\frac{p}{2}-1} + (\omega + t)^q\right).$$

Note that

$$\Phi(\omega) := F(\omega, 0) = \mathcal{A}^{-1}\left(\beta\,(\lambda(\beta) + \beta)\,\omega(\beta^2\omega^2 + |\nabla\omega|^2)^{\frac{p}{2}-1} + \omega^q\right);$$

henceforth any nontrivial fixed point for $\Phi$ would solve problem (3.3).

We have to verify the assumptions of Theorem 3.3. First of all, the compactness of $F(\omega, t)$. If we set $F(\omega, t) = \phi$, then it means that $\phi \in W_0^{1,p}(S)$ satisfies

$$\begin{aligned} -div_g\left((\beta^2\phi^2 + |\nabla\phi|^2)^{\frac{p}{2}-1}\nabla\phi\right) + \beta^2\,\phi(\beta^2\phi^2 + |\nabla\phi|^2)^{\frac{p}{2}-1} \\ = \left(\beta\,(\lambda(\beta) + \beta + t)\,\omega(\beta^2\omega^2 + |\nabla\omega|^2)^{\frac{p}{2}-1} + (\omega + t)^q\right) \end{aligned} \tag{3.6}$$

Thus, if we assume that $\omega$ belongs to a bounded set in $K \cap X$, the right-hand side of (3.6) is bounded in $C(\overline{S})$. Thus, by standard regularity estimates up to the boundary for $p$-Laplace type operators (see [13, Appendix] and [6], [19]), $\phi$ remains bounded in $C^{1,\alpha}(\overline{S})$ and therefore relatively compact in $C^1(\overline{S})$. It remains to show that conditions (i)–(iii) of Theorem 3.3 hold.

*Step 1: Condition (i) holds.* We proceed by contradiction in supposing that there exists a sequence $\{s_n\} \subset [0, 1]$ such that for any $n \in \mathbb{N}$ the following problem

$$\begin{cases} -div_g\left((\beta^2\omega^2 + |\nabla\omega|^2)^{\frac{p}{2}-1}\nabla\omega\right) + \beta^2(\beta^2\omega^2 + |\nabla\omega|^2)^{\frac{p}{2}-1}\omega \\ \qquad = s^{p-1}\beta(\lambda(\beta) + \beta)(\beta^2\omega^2 + |\nabla\omega|^2)^{\frac{p}{2}-1}\omega + s_n^{p-1}\omega^q \qquad \text{in } S \\ \qquad \omega = 0 \quad \text{on } \partial S, \end{cases} \tag{3.7}$$

admits a positive solution $\omega_n$, and that there holds

$$\|\omega_n\|_X \to 0 \quad \text{as } n \to \infty.$$

Set $w_n = \omega_n/\|\omega_n\|$, then $w_n$ solves

$$\begin{cases} -div_g\left((\beta^2 w_n^2 + |\nabla w_n|^2)^{\frac{p}{2}-1}\nabla w_n\right) + \beta^2 w_n(\beta^2 w_n^2 + |\nabla w_n|^2)^{\frac{p}{2}-1} \\ \qquad = s_n^{p-1}\beta(\lambda(\beta) + \beta)(\beta^2 w_n^2 + |\nabla w_n|^2)^{\frac{p}{2}-1}w_n + s_n^{p-1}w_n^q\,\|w_n\|_X^{q-(p-1)} \qquad \text{in } S \\ w_n = 0 \quad \text{on } \partial S \end{cases}$$

Up to subsequences, we assume that $s_n \to s$ for some $s \in [0, 1]$. Using compactness arguments we deduce that $w_n$ will converge strongly in $C^1(\overline{S})$ to some positive function $w$ such that $\|w\|_X = 1$ and which solves

$$\begin{cases} -div_g\left((\beta^2 w^2 + |\nabla w|^2)^{\frac{p}{2}-1}\nabla w\right) \\ \qquad = \beta\left(s^{p-1}\lambda(\beta) + (s^{p-1} - 1)\beta\right)(\beta^2 w^2 + |\nabla w|^2)^{\frac{p}{2}-1}w \quad \text{in } S \\ w = 0 \quad \text{on } \partial S \end{cases} \tag{3.8}$$

Using Theorem 3.1, we derive $\Lambda_\beta = s^{p-1}\lambda(\beta)+(s^{p-1}-1)\beta$. Since $\beta < \beta_S$, $\lambda(\beta) < \Lambda_\beta$ by (3.2). Therefore, as $s \le 1$, we get

$$s^{p-1}\lambda(\beta) + (s^{p-1} - 1)\beta \le s^{p-1}\lambda(\beta) < \Lambda_\beta \,,$$

which is a contradiction. Consequently, there exists $R_1 > 0$ such that for any $s \in [0,1]$, there holds $\omega \ne s\Phi(\omega)$ for any $\omega$ such that $\|\omega\|_X = R_1$.

*Step 2: Condition (ii) holds.* Consider the first eigenvalue $\lambda_{1,\beta}$ associated with the operator $\mathcal{A}$, i.e.

$$\lambda_{1,\beta} = \min\left\{ \int_S (\beta^2\omega^2 + |\nabla\omega|^2)^{\frac{p}{2}}\, dv_g : \omega \in W_0^{1,p}(S)\,,\ \int_S |\omega|^p\, dv_g = 1 \right\} \tag{3.9}$$

Note that for $t$ large enough, we have $\lambda(\beta) + \beta + t \ge 0$, hence, using that $q > p-1$, we can find $T > 0$ such that

$$\beta\,(\lambda(\beta) + \beta + t)\,\omega(\beta^2\omega^2 + |\nabla\omega|^2)^{\frac{p}{2}-1} + (\omega + t)^q \ge (\lambda_1 + \delta)\omega^{p-1} \qquad \forall t \ge T\,, \forall \omega \ge 0\,.$$

Therefore, if $t \ge T$ and $F(\omega, t) = \omega$ we deduce that $\omega \ne 0$ and satisfies

$$\begin{cases} -div_g\left((\beta^2\omega^2 + |\nabla\omega|^2)^{\frac{p}{2}-1}\nabla\omega\right) + \beta^2\omega(\beta^2\omega^2 + |\nabla\omega|^2)^{\frac{p}{2}-1} \ge (\lambda_{1,\beta} + \delta)\omega^{p-1} & \text{in } S \\ \omega = 0 \quad \text{on } \partial S \end{cases}$$

The existence of a positive super-solution with $\lambda_{1,\beta} + \delta$ would make it possible to construct a positive solution as well. But since $\lambda_{1,\beta}$ is an isolated eigenvalue (see Appendix) this yields a contradiction. Therefore, for $t \ge T$ the equation $F(\omega, t) = \omega$ has no solution at all. Note that $T$ only depends on $\lambda_1$, $\beta$.

*Step 3: Condition (iii) holds.* Since we proved that (ii) holds independently on the choice of $R_2$, it is enough to show that (iii) holds for every $t \le T$.

This is done if we have the existence of universal a priori estimates, i.e. if we can prove the existence of a constant $R_2$ such that for any $t \le T$ every positive solution of

$$\begin{cases} -div_g\left((\beta^2\omega^2 + |\nabla\omega|^2)^{\frac{p}{2}-1}\nabla\omega\right) + \beta^2\omega(\beta^2\omega^2 + |\nabla\omega|^2)^{\frac{p}{2}-1} = \\ \qquad \beta(\lambda(\beta) + \beta + t)(\beta^2\omega^2 + |\nabla\omega|^2)^{\frac{p}{2}-1}\omega + (\omega + t)^q \quad \text{in } S \\ \qquad \omega = 0 \quad \text{on } \partial S \end{cases}$$

satisfies $\|\omega\| < R_2$.

The crucial step is to prove that there exist universal a priori estimates for the $L^\infty$-norm (a bound for the $W_0^{1,p}$-norm would follow immediately, and then a bound in $X$ from the regularity theory). A standard procedure is to reach this result reasoning by contradiction and using a blow-up argument. Indeed, if a universal bound does not exist, there exist a sequence of solutions $\omega_n$ and $t_n \le T$ such that

$$\|\omega_n\|_\infty \to \infty\,.$$

Let $\sigma_n$ be the (local coordinates of) maximum points of $\omega_n$; up to subsequences, we have $\sigma_n \to \sigma_0 \in \overline{S}$. Setting $M_n = \|\omega_n\|_\infty^{-\frac{q-(p-1)}{p}}$, define

$$v_n(y) = \frac{\omega_n(\sigma_n + M_n y)}{\|\omega_n\|_\infty} = M_n^{\frac{p}{q-(p-1)}}\,\omega_n(\sigma_n + M_n y)$$

Then $v_n$ is a sequence of uniformly bounded solutions, which will be locally compact in the $C^1$-topology. Rescaling the equation and passing to the limit in $n$ we find out that the limit function $v$ is positive and satisfies the equation

$$-\Delta_p v = c_0 v^q$$

for some constant $c_0$ (coming out from the local expression of Laplace-Beltrami operator). Depending whether $\sigma_0 \in S$ or $\sigma_0 \in \partial S$, the equation would take place in either $\mathbb{R}^d$ or in the half space $\mathbb{R}^d_+$, where $d = N-1$, in which case $v$ vanishes on $\partial\mathbb{R}^d_+$. Since $p-1 < q < q_c$, this contradicts either Theorem 3.4, or Theorem 3.5 because, by construction, we have $v(0) = 1$. $\square$

*Remark.* In the case $p = 2$, existence is proved in [3] using a standard variational method. It is also proved that, if $(M, g) = (S^d, g_0)$ (the standard sphere), and if $S$ is a spherical cap with center $a$, any positive solution of

$$\begin{cases} \Delta'\omega + \beta(\beta + 1 - d))\omega + \omega^q = 0 & \text{in } S \\ \omega = 0 & \text{on } \partial S, \end{cases} \tag{3.10}$$

depends only on the angle $\theta$ from $a$. Furthermore, uniqueness is proved by a delicate analysis of the non-autonomous second order O.D.E. satisfied by $\omega$. In the case $p \neq 2$ and assuming always that $S$ is a spherical cap of $(S^d, g_0)$, it is still possible to construct a radial (i.e. depending only on $\theta$) positive solution of (3.3): it suffices to restrict the functional analysis framework to radial functions. However, there are two interesting open questions the answer to which would be important:

(i) *Are all positive solutions of* (*3.3*) *radial ?*

(ii) *Is there uniqueness of positive radial solutions of* (*3.3*)*?*

## 4 Existence for the absorption problem

Let us now consider the absorption problem, namely (1.8) with $\epsilon = -1$. We give an existence result which extends the previous ones obtained in [20], with a simpler proof.

**Theorem 4.1** *Assume $0 < p-1 < q$. Then for any $\beta > \beta_S$, there exists a unique positive function $\omega \in C(\overline{S}) \cap C^2(S)$ satisfying*

$$\begin{cases} -div_g\left((\beta^2\omega^2 + |\nabla\omega|^2)^{\frac{p}{2}-1}\nabla\omega\right) = \beta\lambda(\beta)(\beta^2\omega^2 + |\nabla\omega|^2)^{\frac{p}{2}-1}\omega - \omega^q & \text{in } S \\ \omega = 0 \quad \text{on } \partial S, \end{cases} \tag{4.1}$$

*where $\lambda(\beta) = \beta(p-1) + p - d - 1$.*

Before proving Theorem 4.1, we will need the following lemma.

**Lemma 4.2** *For $\beta > 0$ and $p > 1$, let $\Lambda_\beta$ and $\beta_S$ be defined by Theorem 3.1. Then both $\Lambda_\beta$ and $\beta_S$ are continuous functions of $p$, varying in $(1, \infty)$.*

*Proof.* By Theorem 3.1, $\Lambda_\beta$ is uniquely defined for any fixed $p > 1$. To emphasize the dependence of $\Lambda_\beta$ on $p$, let us denote it now by $\Lambda_{\beta,p}$. The continuity of $\Lambda_{\beta,p}$ with respect to $p$ can be proved in the same way as we proved (see Proposition 2.4 in [13]) the continuity of $\Lambda_{\beta,p}$ with respect to $\beta$. Thus, we only sketch the argument, which relies on the construction itself of $\Lambda_{\beta,p}$. Indeed, we proved in [13] that $\Lambda_{\beta,p}$ is the unique constant such that there exists a function $v \in C^2(S)$ satisfying

$$\begin{cases} -\Delta_g v - (p-2)\dfrac{D^2 v \nabla v . \nabla v}{1 + |\nabla v|^2} + \beta(p-1)|\nabla v|^2 = -\Lambda_{\beta,p} & \text{in } S \\ \lim\limits_{\sigma \to \partial S} v(\sigma) = \infty. & \end{cases} \tag{4.2}$$

If we normalize $v$ by setting, for example, $v(\sigma_0) = 0$ for some $\sigma_0 \in S$, then $v$ is unique. Moreover $v \in C^2(S)$ and $v$ satisfies estimates in $W^{1,\infty}_{loc}(S)$ which are uniform as $\beta \in (0, \infty)$ and $p \in (1, \infty)$ vary in compact sets. It is also easy to check (see [13]) that $\Lambda_{\beta,p}$ remains bounded whenever $\beta$ varies in a compact set of $(0, \infty)$ and $p$ vary in a compact set of $(1, \infty)$. The estimates obtained on $v$ and $\nabla v$ imply that, whenever $\beta_n$ or $p_n$ are convergent sequences, the sequence of corresponding solutions $v_n$ of (4.2) (such that $v_n(\sigma_0) = 0$) is relatively compact (locally uniformly in $C^1$). The equation (4.2) turns out then to be stable (including the boundary estimates); finally, the uniqueness property of $\Lambda_{\beta,p}$, and of the associated (normalized) solution $v$, implies the continuity of $\Lambda_{\beta,p}$ with respect to both $\beta$ and $p$.

Let now $\beta_{S,p}$ be the spectral exponent defined by the equation

$$\Lambda_{\beta,p} = \beta(p-1) + p - d - 1 \tag{4.3}$$

First of all note that when $p$ lies in a compact set in $(1, \infty)$, then necessarily $\beta_{S,p}$ is bounded. Indeed, since $\Lambda_{\beta,p} \le \Lambda_{1,p}$ whenever $\beta \ge 1$, we have that

$$\beta_S(p-1) + p - d - 1 \le \Lambda_{1,p} \quad \text{if } \beta_S \ge 1,$$

so that

$$\beta_S \le 1 + \frac{1}{p-1}\left(\Lambda_{1,p} - (p-d-1)\right) .$$

Therefore, if $p$ belongs to a compact set in $(1, \infty)$, then $\beta_S$ remains also in a bounded set. Now, if $p_n \to p_0$, setting $\beta_n = \beta_{S,p_n}$, we have that $\beta_n$ is bounded and, up to subsequences, it is convergent to some $\beta_0$. From (4.3), we deduce that $\Lambda_{\beta_n,p_n}$ is bounded, which implies that $\beta_n$ cannot converge to zero, hence $\beta_0 > 0$. Then, using the continuity of $\Lambda_{\beta,p}$, we can pass to the limit in (4.3) and we deduce that $\beta_0$ is the spectral exponent with $p = p_0$, i.e. $\beta_0 = \beta_{S,p_0}$. This proves that $\beta_{S,p}$ is continuous with respect to $p$. □

We are now ready to prove Theorem 4.1.

*Proof of Theorem 4.1.*

*Step 1: construction of a solution.* We use similar ideas as in the proof of Theorem 3.2, i.e. a topological degree argument. On the Banach space $X = C_0^1(\overline{S})$ (endowed with its natural norm) with positive cone $K$, we set

$$\mathcal{B}(\omega) = -div_g\left((\beta^2\omega^2 + |\nabla\omega|^2)^{\frac{p}{2}-1}\nabla\omega\right) + \beta^2(\beta^2\omega^2 + |\nabla\omega|^2)^{\frac{p}{2}-1}\omega + |\omega|^{q-1}\omega \quad (4.4)$$

$$\Psi(\omega) := \mathcal{B}^{-1}\left(\beta\left(\lambda(\beta) + \beta\right)(\beta^2\omega^2 + |\nabla\omega|^2)^{\frac{p}{2}-1}\omega_+\right).$$

Clearly, $\Psi(w) = w$ implies that $w \geq 0$ and solves (4.1). Then, it is enough to prove the existence of a non trivial fixed point for $\Psi$. Observe that, as in Theorem 3.2, $\Psi$ is a continuous compact operator in $X$ thanks to the $C^{1,\alpha}$ estimates for $p$–Laplace operators, and $\Psi(K) \subset K$.

We now wish to compute the degree of $I - \Psi$. First of all we consider, if $R$ is sufficiently large, $\deg(I - \Psi, B_R^+, 0)$ where $B_R^+ = B_R \cap K$. To this purpose, define, for $t \in [0, 1]$, $\Psi^*(\omega, t) = t\Psi(\omega)$. Then $\Psi^*$ is a compact map on $X \times [0, 1]$ and if $\Psi^*(\omega, t) = \omega$, we have

$$\begin{aligned}-div_g\left((\beta^2\omega^2 + |\nabla\omega|^2)^{\frac{p}{2}-1}\nabla\omega\right) + \beta^2(\beta^2\omega^2 + |\nabla\omega|^2)^{\frac{p}{2}-1}\omega + \tfrac{1}{t^{q-(p-1)}}\omega^q \\ = t^{p-1}\beta\left(\lambda(\beta) + \beta\right)(\beta^2\omega^2 + |\nabla\omega|^2)^{\frac{p}{2}-1}\omega.\end{aligned} \quad (4.5)$$

We get, by the maximum principle,

$$\left\|\frac{\omega}{t}\right\|_\infty^{q-(p-1)} \leq t^{p-1}\beta^{p-1}\left(\lambda(\beta) + \beta\right) \leq \beta^{p-1}\left(\lambda(\beta) + \beta\right).$$

Since $t \leq 1$, we deduce in particular that $\|\omega\|_\infty$ is bounded independently on $t$. Then, we have

$$\frac{1}{t^{q-(p-1)}}\omega^q \leq \left\|\frac{\omega}{t}\right\|_\infty^{q-(p-1)} \|\omega\|_\infty^{p-1} \leq C\|\omega\|_\infty^{p-1} \leq C.$$

Multiplying by $\omega$ we obtain a similar bound for $\|\omega\|_{W_0^{1,p}(S)}$, and the regularity theory for $p$–Laplace type equations yields a further estimate on $\|\nabla\omega\|_\infty$. Therefore, we conclude that there exists a constant $M$, independent on $t \in [0, 1]$, such that $t\Psi(\omega) = \omega$ implies $\|\omega\|_X \leq M$. As a consequence, if $R$ is sufficiently large we have $t\Psi(\omega) \neq \omega$ on $\partial B_R$. We deduce that $\deg(I - t\Psi, B_R^+, 0)$ is constant. Therefore

$$\deg(I - \Psi, B_R^+, 0) = \deg(I - t\Psi, B_R^+, 0) = \deg(I, B_R^+, 0) = 1. \quad (4.6)$$

Next, we compute $\deg(I - \Psi, B_r^+, 0)$ for small $r$. We set

$$\mathcal{B}_t(\omega) = -div_g\left((\beta^2\omega^2 + |\nabla\omega|^2)^{\frac{p}{2}-1}\nabla\omega\right) + \beta^2(\beta^2\omega^2 + |\nabla\omega|^2)^{\frac{p}{2}-1}\omega + t|\omega|^{q-1}\omega \quad (4.7)$$

and
$$F(\omega,t) := \mathcal{B}_t{}^{-1}\left(\beta\left(\lambda(\beta)+\beta\right)\omega_+(\beta^2\omega^2+|\nabla\omega|^2)^{\frac{p}{2}-1}\right).$$
Again, we have $\Psi(\cdot) = F(\cdot,1)$. We claim that there exists a small $r > 0$ such that $F(\omega,t) \neq \omega$ for every $t \in [0,1]$ and $\omega \in \partial B_r$. Indeed, reasoning by contradiction, if this were not true there would exist a nonnegative sequence $\omega_n$ such that $0 \neq \|\omega_n\| \to 0$, and $t_n \in [0,1]$ such that $F(\omega_n,t_n) = \omega_n$, which means that
$$\begin{aligned}-div_g\left((\beta^2\omega_n^2+|\nabla\omega_n|^2)^{\frac{p}{2}-1}\nabla\omega_n\right) + \beta^2(\beta^2\omega_n^2+|\nabla\omega_n|^2)^{\frac{p}{2}-1}\omega_n + t_n\omega_n^q \\ = \beta\left(\lambda(\beta)+\beta\right)\omega_n(\beta^2\omega_n^2+|\nabla\omega_n|^2)^{\frac{p}{2}-1}\end{aligned}$$
Dividing by $\|\omega_n\|^{p-1}$ and letting $n\to\infty$, we find that $\frac{\omega_n}{\|\omega_n\|}$ would converge to some function $\hat{w}$ such that $\hat{w} \geq 0$, $\|\hat{w}\| = 1$ and
$$\begin{aligned}-div_g\left((\beta^2\hat{\omega}^2+|\nabla\hat{\omega}|^2)^{\frac{p}{2}-1}\nabla\hat{\omega}\right) + \beta^2(\beta^2\hat{\omega}^2+|\nabla\hat{\omega}|^2)^{\frac{p}{2}-1}\hat{\omega} \\ = \beta\left(\lambda(\beta)+\beta\right)\hat{\omega}(\beta^2\hat{\omega}^2+|\nabla\hat{\omega}|^2)^{\frac{p}{2}-1}\end{aligned}$$
By Theorem 3.1 this means that $\lambda(\beta) = \Lambda_\beta$, which is not possible since $\lambda(\beta) > \Lambda_\beta$ because $\beta > \beta_S$ (see Remark 3.1). Therefore, we conclude that $F(\omega,t) \neq \omega$ for every $t \in [0,1]$ and $\omega \in \partial B_r$ provided $r$ is sufficiently small. We deduce that $\deg(I - F(\cdot,t), B_r, 0)$ is constant and in particular
$$\deg(I-\Psi, B_r^+, 0) = \deg(I - F(\cdot,0), B_r^+, 0).$$
In order to compute this degree, we perform an homotopy acting on $p$ and $\beta$ by setting $p_t = 2t + (1-t)p$ and by taking $\beta_t$ so that $t \mapsto \beta_t$ is continuous on $[0,1]$, $\beta_0 = \beta$, $\beta_t > \beta_{S,p_t}$ for every $t \in [0,1]$ (where $\beta_{S,p_t}$ is the spectral exponent for $S$ with $p = p_t$) and $\beta_1 > 0$ is large enough. It follows from Lemma 4.2 that $\beta_{S,p_t}$ is a continuous function of $t$ and remains bounded as $t \in [0,1]$. Therefore, a similar choice of function $\beta_t$ is possible. In the space $C_0^1(\overline{S})$ we define the mapping $\mathcal{C}_t$ by
$$\mathcal{C}_t(\omega) := -div_g\left((\beta_t^2\omega^2+|\nabla\omega|^2)^{\frac{p_t}{2}-1}\nabla\omega\right) + \beta_t^2(\beta_t^2\omega^2+|\nabla\omega|^2)^{\frac{p_t}{2}-1}\omega. \tag{4.8}$$
We set
$$\tilde{F}(\omega,t) = \mathcal{C}_t^{-1}\left(\beta_t(\lambda(\beta_t)+\beta_t)(\beta_t^2\omega^2+|\nabla\omega|^2)^{\frac{p_t}{2}-1}\omega\right). \tag{4.9}$$
Combining the Tolksdorf's construction [19] which shows the uniformity with respect to $p_t$ of the $C^{1,\alpha}$ estimates (with $\alpha = \alpha_t \in (0,1)$), with the perturbation method of [13, Th A1], we obtain that $(\omega,t) \mapsto \tilde{F}(\omega,t)$ is compact in $C_0^1(\overline{S}) \times [0,1]$. Since $\beta_t > \beta_{S,p_t}$, clearly $I - \tilde{F}(.,t)$ does not vanish on $\|\omega\|_X = r$ for any $r > 0$ which implies that
$$\deg(I-\Psi, B_r^+, 0) = \deg(I-\tilde{F}(\cdot,0), B_r^+, 0) = \deg(I-\tilde{F}(\cdot,1), B_r^+, 0).$$
But
$$I - \tilde{F}(\cdot,1) = I - \beta_1(\lambda(\beta_1)+\beta_1)(-\Delta_g+\beta_1^2)^{-1}. \tag{4.10}$$

Since $-\Delta_g$ has only one eigenvalue in $S$ with positive eigenfunction and multiplicity one, choosing $\beta_1$ large in a way that $\lambda(\beta_1)\beta_1 > \lambda_1(S)$ it follows that

$$\deg(I - \tilde{F}(\cdot, 1), B_r^+, 0) = -1 = \deg(I - \Psi, B_r^+, 0)\,.$$

To conclude, since we have

$$\deg(I - \Psi, B_R^+ \setminus \overline{B}_r^+, 0) = \deg(I - \Psi, B_R^+, 0) - \deg(I - \Psi, B_r^+, 0) \neq 0$$

we deduce the existence of some $\omega$ such that $r < \|\omega\| < R$ which is a solution of (4.1).

*Step 2: uniqueness.* If $\omega$ is any positive solution, then $\beta^2\omega^2 + |\nabla\omega^2|$ is positive in $\overline{S}$. This is obvious in $S$ and it is a consequence of Hopf boundary lemma on $\partial S$. Let $\varpi$ and $\omega$ be two positive solutions. Either the two functions are ordered or their graphs intersect. Since all the solutions are positive in $S$ and satisfy Hopf boundary lemma, we can define

$$\theta := \inf\{s \geq 1 : s\omega \geq \varpi\},$$

and denote $\omega^* := \theta\omega$. Either the graphs of $\varpi$ and $\omega^* := \theta\omega$ are tangent at some interior point $\alpha \in S$, or $\omega^* > \varpi$ in $S$ and there exists $\alpha \in \partial S$ such that $\varpi_\nu(\alpha) = \omega^*_\nu(\alpha) < 0$. We put $w = \varpi - \omega^*$ and use local coordinates $(\sigma_1, ..., \sigma_d)$ on $M$ near $\alpha$. We denote by $g = (g_{ij})$ the metric tensor on $M$ and $g^{jk}$ its contravariant components. Then, for any $\varphi \in C^1(S)$,

$$|\nabla\varphi|^2 = \sum_{j,k} g^{jk} \frac{\partial\varphi}{\partial\sigma_j}\frac{\partial\varphi}{\partial\sigma_k} = \langle\nabla\varphi, \nabla\varphi\rangle_g.$$

If $X = (X^1, ...X^d) \in C^1(TM)$ is a vector field, if we lower indices by setting $X^\ell = \sum_i g^{\ell i} X_i$, then

$$div_g X = \frac{1}{\sqrt{|g|}} \sum_\ell \frac{\partial}{\partial\sigma_\ell}\left(\sqrt{|g|}X^\ell\right) = \frac{1}{\sqrt{|g|}} \sum_{\ell,i} \frac{\partial}{\partial\sigma_\ell}\left(\sqrt{|g|}g^{\ell i}X_i\right).$$

By the mean value theorem applied to

$$t \mapsto \Phi(t) = \left(\beta^2(\omega^* + tw)^2 + |\nabla(\omega^* + tw)|^2\right)^{(\frac{p}{2}-1)}(\omega^* + tw) \qquad t = 0, 1,$$

we have, for some $t \in (0,1)$,

$$(\beta^2\varpi^2 + |\nabla\varpi|^2)^{(\frac{p}{2}-1)}\varpi - (\beta^2\omega^{*2} + |\nabla\omega^*|^2)^{(\frac{p}{2}-1)}\omega^* = \sum_j a_j \frac{\partial w}{\partial\sigma_j} + bw,$$

where

$$b = \left(\beta^2(\omega^* + tw)^2 + |\nabla(\omega^* + tw)|^2\right)^{(\frac{p}{2}-2)}\left((p-1)\beta^2(\omega^* + tw)^2 + |\nabla(\omega^* + tw)|^2\right)$$

and

$$a_j = (p-2)\left(\beta^2(\omega^*+tw)^2+|\nabla(\omega^*+tw)|^2\right)^{(\frac{p}{2}-2)}(\omega^*+tw)\sum_k g^{jk}\frac{\partial(\omega^*+tw)}{\partial\sigma_k}$$

Considering now

$$t\mapsto\Phi_i(t)=\left(\beta^2(\omega^*+tw)^2+|\nabla(\omega^*+tw)|^2\right)^{(\frac{p}{2}-1)}\frac{\partial(\omega^*+tw)}{\partial\sigma_i}\qquad t=0,1,$$

we see that there exists some $t_i\in(0,1)$ such that

$$(\beta^2\overline{\omega}^2+|\nabla\overline{\omega}|^2)^{(\frac{p}{2}-1)}\frac{\partial\overline{\omega}}{\partial\sigma_i}-(\beta^2\omega^{*2}+|\nabla\omega^*|^2)^{(\frac{p}{2}-1)}\frac{\partial\omega^*}{\partial\sigma_i}=\sum_j a_{ij}\frac{\partial w}{\partial\sigma_j}+b_iw,$$

where

$$b_i=(p-2)\left(\beta^2(\omega^*+t_iw)^2+|\nabla(\omega^*+t_iw)|^2\right)^{(\frac{p}{2}-2)}\beta^2(\omega^*+t_iw)\frac{\partial(\omega^*+t_iw)}{\partial\sigma_i}$$

and

$$a_{ij}=(p-2)\left(\beta^2(\omega^*+t_iw)^2+|\nabla(\omega^*+t_iw)|^2\right)^{(\frac{p}{2}-2)}\frac{\partial(\omega^*+t_iw)}{\partial\sigma_i}\sum_k g^{jk}\frac{\partial(\omega^*+t_iw)}{\partial\sigma_k}$$
$$+\delta_i^j\left(\beta^2(\omega^*+t_iw)^2+|\nabla(\omega^*+t_iw)|^2\right)^{(\frac{p}{2}-1)}.$$

Set $P=\omega^*(\alpha)=\overline{\omega}(\alpha)$ and $Q=\nabla\omega^*(\alpha)=\nabla\overline{\omega}(\alpha)$. Then $P^2+|Q|^2>0$ and

$$b_i(\alpha)=(p-2)\left(\beta^2P^2+|Q|^2\right)^{(\frac{p}{2}-2)}\beta^2\,PQ_i,$$

and

$$a_{ij}(\alpha)=\left(\beta^2P^2+|Q|^2\right)^{\frac{p}{2}-2}\left(\delta_i^j(\beta^2P^2+|Q|^2)+(p-2)Q_i\sum_k g^{jk}Q_k\right).$$

Because $\omega^*$ is a supersolution for (4.1), the function $w$ satisfies

$$-\frac{1}{\sqrt{|g|}}\sum_{\ell,j}\frac{\partial}{\partial\sigma_\ell}\left(A_{j\ell}\frac{\partial w}{\partial\sigma_j}\right)+\sum_i C_i\frac{\partial w}{\partial\sigma_i}+Dw\le 0\tag{4.11}$$

where the $C_i$ and $D$ are continuous functions and

$$A_{j\ell}=\sqrt{|g|}\sum_i g^{\ell i}a_{ij}.$$

The matrix $(a_{ij})(a)$ is symmetric definite and positive since it is the Hessian of

$$x=(x_1,...,x_d)=\frac{1}{p}(P^2+|x|^2)^{\frac{p}{2}}=\frac{1}{p}\left(P^2+\sum_{j,k}g^{jk}x_jx_k\right)^{\frac{p}{2}}$$

Therefore the matrix $(A_{j\ell})$ keeps the same property in a neighborhood of $a$. Since $w$ is nonpositive and vanishes at some $a\in S$ or $w<0$ and $w_\nu=0$ at some boundary point, it follows from the strong maximum principle or Hopf boundary lemma (see [14]) that $w\equiv 0$, i.e. $\theta\omega=\overline{\omega}$. This implies that actually $\theta=1$ and $\omega=\overline{\omega}$. $\square$

# 5 Appendix

We prove here the following result

**Theorem 5.1** *Let $S$ be a subdomain of a complete $d$-dimensional Riemannian manifold $(M,g)$. If $\beta > 0$ and $p > 1$, the first eigenvalue $\lambda_{1,\beta}$ of the operator $\omega \mapsto -div((\beta^2\omega^2 + |\nabla\omega|^2)^{\frac{p}{2}-1}\nabla\omega) + \beta^2\omega(\beta^2\omega^2 + |\nabla\omega|^2)^{\frac{p}{2}-1}$ in $W_0^{1,p}(S)$ is isolated. Furthermore any corresponding eigenfunction has constant sign.*

*Proof.* The proof is an adaptation of the original one due to Anane and Lindqvist when $\beta = 0$. We recall that

$$\lambda_{1,\beta} = \inf\left\{\int_S (\beta^2\omega^2 + |\nabla\omega|^2)^{\frac{p}{2}} dv_g : \omega \in W_0^{1,p}(S), \int |\omega|^p dv_g = 1\right\}, \tag{5.1}$$

and that there exists $\omega \in W_0^{1,p}(S) \cap C^{1,\alpha}(S)$ such that

$$-div((\beta^2\omega^2 + |\nabla\omega|^2)^{\frac{p}{2}-1}\nabla\omega) + \beta^2\omega(\beta^2\omega^2 + |\nabla\omega|^2)^{\frac{p}{2}-1} = \lambda_{1,\beta}|\omega|^{p-2}\omega \quad \text{in } S. \tag{5.2}$$

The function $|\omega|$ is also a minimizer for $\lambda_{1,\beta}$, thus it is a positive solution of (5.2). By Harnack inequality [16], for any compact subset $K$ of $S$, there exists $C_K$ such that

$$\frac{|\omega|(\sigma_1)}{|\omega|(\sigma_2)} \le C_K \qquad \forall \sigma_i \in K,\ i = 1,2.$$

Thus any minimizer $\omega$ must keep a constant sign in $S$. If $\lambda_{1,\beta}$ is not isolated, there exists a decreasing sequence $\{\mu_n\}$ of real numbers converging to $\lambda_{1,\beta}$ and a sequence of functions $\omega_n \in W_0^{1,p}(S)$, solutions of

$$-div(\beta^2\omega_n^2 + |\nabla\omega_n|^2)^{\frac{p}{2}-1}\nabla\omega_n) + \beta^2\omega_n(\beta^2\omega_n^2 + |\nabla\omega_n|^2)^{\frac{p}{2}-1} = \mu_n|\omega_n|^{p-2}\omega_n \quad \text{in } S \tag{5.3}$$

such that $\|\omega_n\|_{L^p(S)} = 1$. By standard compactness and regularity results, we can assume that $\omega_n \to \overline{\omega}$ weakly in $W_0^{1,p}(S)$ and strongly in $L^p(S)$. Thus

$$\int_S (\beta^2\overline{\omega}^2 + |\nabla\overline{\omega}|^2)^{\frac{p}{2}} dv_g \le \liminf_{n\to\infty} \int_S (\beta^2\omega_n^2 + |\nabla\omega_n|^2)^{\frac{p}{2}} dv_g = \lambda_{1,\beta}$$

which implies that $\overline{\omega}$ is an eigenfunction associated with $\lambda_{1,\beta}$.

We observe that $\omega_n$ cannot have constant sign. Indeed, if we had that $\omega_n$ is positive in $\Omega$, we could proceed as in the proof of Theorem 4.1-Step 2; up to rescaling $\omega_n$, we could assume that $w = \omega - \omega_n$ is nonpositive, is not zero, and the graphs of $\omega$ and $\omega_n$ are tangent. In that case, using (5.2) and (5.3), we see that $w$ satisfies a nondegenerate elliptic equation (as in (4.11)), and we obtain a contradiction either by the strict maximum principle or by Hopf lemma. Thus, any eigenfunction $\omega_n$ must change sign in $\Omega$. Set $S_n^+ = \{\sigma \in S : \omega_n(\sigma) > 0\}$ and $S_n^- = \{\sigma \in S : \omega_n(\sigma) < 0\}$. Clearly, for $0 < \theta < 1$,

$$\int_{S_n^\pm} (\beta^2\omega_n^2 + |\nabla\omega_n|^2)^{\frac{p}{2}} dv_g \ge (1-\theta)\beta^p \int_{S_n^\pm} |\omega_n|^p dv_g + \theta \int_{S_n^\pm} |\nabla\omega_n|^p dv_g.$$

It follows from (5.3), multiplying by $\omega_n^+$, that

$$\int_{S_n^+}(\beta^2\omega_n^2+|\nabla\omega_n|^2)^{\frac{p}{2}}dv_g=\mu_n\int_{S_n^+}|\omega_n|^p dv_g$$

hence

$$\mu_n\int_{S_n^+}|\omega_n|^p dv_g\geq(1-\theta)\beta^p\int_{S_n^+}|\omega_n|^p dv_g+\theta\int_{S_n^+}|\nabla\omega_n|^p dv_g.$$

Since for some suitable $q>p$ (for example $q=p^*$ if $p<d$, or any $p<q<\infty$ if $p\geq d$)

$$\int_{S_n^+}|\nabla\omega_n|^p dv_g\geq c(p,q)\left(\int_{S_n^+}|\omega_n|^q dv_g\right)^{\frac{p}{q}}\geq c(p,q)|S_n^+|^{\frac{p-q}{q}}\int_{S_n^+}|\omega_n|^p dv_g$$

we obtain

$$\mu_n\geq(1-\theta)\beta^p+\theta c(p,q)|S_n^+|^{\frac{p-q}{q}}.$$

Similarly we get, multiplying (5.3) by $\omega_n^-$, that

$$\mu_n\geq(1-\theta)\beta^p+\theta c(p,q)|S_n^-|^{\frac{p-q}{q}}.$$

It follows that the two sets

$$S^{\pm}=\limsup_{n\to\infty}S_n^{\pm}$$

have positive measure. Since $\overline{\omega}\geq 0$ on $S^+$ and $\overline{\omega}\leq 0$ on $S^-$, we derive a contradiction with the fact that any eigenfunction corresponding to $\lambda_{1,\beta}$ has constant sign. $\square$

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